Sur des classes de fonctions à seuil caractérisables par des contraintes relationnelles

On classes of threshold functions characterizable by relational constraints

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Résumé :

Motivés par les sémantiques modales induites par des jeux de majorité, nous considerons la classe des fonctions à seuil. L. Hellerstein a montré que cette classe peut être caractérisée par des contraintes relationnelles (ou, de façon équivalente, par des équations fonctionnelles), mais aussi qu'il faut un nombre infini de ces contraintes pour la caractériser. Dans cet article, nous présentons une classification complète des classes de fonctions à seuil induites par des clones booléens, en identifiant ceux qui ont une caractérisation finie. De plus, nous présentons les ensembles des contraintes relationnelles qui caractérisent chacune de ces classes.

Mots-clés :

Fonction booléenne, fonction à seuil, clone, caractérisation, contrainte relationnelle, équation fonctionnelle.

Abstract:

Motivated by modal semantics induced by majority games, we consider the class of threshold functions. It was shown by L. Hellerstein that this class is characterizable by relational constraints (or equivalently, by functional equations), but that there is no characterization by means of finitely many constraints. In this paper, we present a complete classification of classes of threshold functions induced by Boolean clones, into whether they are characterizable by finitely many relational constraints. Moreover we provide sets of constraints characterizing each of such classes.

Keywords:

Boolean function, threshold function, clone, characterization, relational constraint, functional equation

1 Introduction and preliminaries

1.1 Introduction

Two approaches to characterize properties of Boolean functions have been considered recently: one in terms of functional equations [9], another in terms of relational constraints [19]. As it turns out, these two approaches have the same expressive power in the sense that they characterize the same properties (classes) of Boolean functions, which can be described as initial segments of the so-called "minor" relation between functions: for two functions f and g of several variables, f is said to be a minor of g if f can be obtained from g by identifying, permuting or adding inessential variables (see Subsection 1.3). Furthermore, a class is characterizable by a finite number of functional equations if and only if it is caracterizable by a finite number of relational constraints.

Apart from the theoretical interest, these approaches were shown in [4] to be tightly related to frame definability within modal logic, and a complete correspondence between classes of Boolean functions and classes of Scott-Montague frames $\langle W, F\rangle$, where W is a finite set and $F\colon \mathcal{P}(W) \to \mathcal{P}(W)$, for modal logic was established. This correspondence is based on the natural bijection between maps $F\colon \mathcal{P}(W) \to \mathcal{P}(W)$ and vector-valued Boolean functions $f\colon \mathbb{B}^{|W|} \to \mathbb{B}^{|W|}$. For basic background on modal logic see, e.g., [1].

An attractive aspect of this correspondence is that equational theories (or, equivalently, constraint theories) of classes of Boolean functions translate straightforwardly into theories of the corresponding classes of modal frames. This setting was refined to several correspondences between important classes of Boolean functions (such as clones and other noteworthy equational classes) and classes of Kripke-like structures by considering several variants to classical modal semantics.

Motivated by modal semantics based on majority games, in this paper we consider classes of threshold functions, that is, Boolean functions that have the property that the true points can be separated from the false points by a hyperplane when considered as elements of the n-dimensional real space $\mathbb{R}^n$. Such functions have been widely studied in the existing literature on Boolean functions, switching theory, system reliability theory, game theory, etc.; for background see, e.g., [12, 16, 17, 18, 20, 22].

This property is known to be expressible by relational constraints (or equivalently, by functional equations) but no finite set of such objects is able to capture this property (see [11]). However, by imposing additional conditions such as linearity or preservation of componentwise conjunctions or disjunctions of tuples, the resulting classes of threshold functions may become characterizable by a finite number of relational constraints. In fact, these examples can be obtained from the class of threshold functions by intersecting it with certain clones (i.e., classes of functions containing all projections and closed under functional composition), namely, those of linear functions, conjunctions and disjunctions, respectively. Another noteworthy and well-known example of such an intersection is the class of "majority games", which results as the intersection with the clone of self-dual monotone functions. The natural question is then: Is the class of majority games characterizable by a finite number of relational constraints?

In this paper we answer negatively to this question by determining which intersections $T \cap C$ of the class T of threshold functions with a clone C are characterizable by a finite set of relational constraints. Moreover, we provide finite or infinite characterizing sets of relational constraints accordingly. Such characterizing sets can then be used to axiomatize classes of "weighted median Kripke frames" that can account for modal semantics induced by majority games. Essentially, a *weighted median Kripke frame* is a structure $\mathbf{K} = \langle W, D \rangle$ where W is a nonempty finite set and D is a function $W^2 \to \mathbb{N}$ satisfying the condition: for each $w \in W$, the sum $\sum_{v \in W} D(w, v)$ is odd. A *weighted median Kripke model* is then a structure $\mathbf{M} = \langle \mathcal{K}, V \rangle$ where $\mathbf{K} = \langle W, D \rangle$ is a weighted median Kripke frames and V is a valuation $\Phi \to \mathcal{P}(W)$. In these models, the truth-value of propositional formulas is given as usual but the truth-value of modal formulas of the form $\Box\phi$ is given by: $\mathbf{M}, w \models \Box\phi$ if and only if

$$\sum_{v \in \|\phi\|^{\mathbf{M}}} D(w, v) \geq \frac{\sum_{v \in W} D(w, v) + 1}{2},$$

where $\|\phi\|^{\mathbf{M}} := \{v \in W : \mathbf{M}, v \models \phi\}$.

This model-theoretic approach to modal logic was considered by Virtanen in [21] where the basic modal language is used to reason about knowledge and belief. Here, Virtanen proposes a variation of epistemic logic and introduces a model-theoretic approach in which weights represent probabilities of possible events.

Given the page limit for this contribution, we will only focus on the former aspect, namely, constraint characterizations of classes of threshold functions. The latter aspect is the subject of a manuscript being prepared in collaboration between the authors and L. Hella and J. Kivelä at the University of Tampere.

The paper is organized as follows. In the remainder of this section, we recall basic notions and results that will be needed throughout the paper. The main results are presented in Section 2, namely, the classification of all intersections $C \cap T$ as well as the corresponding characterizing sets of relational constraints. The Appendix provides further background on the theory of Boolean clones.

1.2 Boolean functions

Throughout the paper, we denote the set $\{1, \dots, n\}$ by $[n]$ and the set $\{0, 1\}$ by $\mathbb{B}$.

We will denote tuples in $\mathbb{B}^m$ by boldface letters and their entries with corresponding italic letters, e.g., $\mathbf{a} = (a_1, \dots, a_m)$. Tuples $\mathbf{a} \in \mathbb{B}^m$ may be viewed as mappings $\mathbf{a} \colon [m] \to \mathbb{B}$, $i \mapsto a_i$. With this convention, given a map $\sigma \colon [n] \to [m]$, we can write the tuple $(a_{\sigma(1)}, \dots, a_{\sigma(n)})$ as $\mathbf{a} \circ \sigma$, or simply $\mathbf{a}\sigma$.

A *Boolean function* is a map $f \colon \mathbb{B}^n \to \mathbb{B}$ for some positive integer n called the *arity* of f. Typical examples of Boolean functions include

- the n-ary i-th *projection* ($i \in [n]$) $e_i^{(n)} \colon \mathbb{B}^n \to \mathbb{B}$, $(a_1, \dots, a_n) \mapsto a_i$;
- the n-ary 0-*constant* and 1-*constant* functions $\mathbf{0}^{(n)}, \mathbf{1}^{(n)} \colon \mathbb{B}^n \to \mathbb{B}$, $\mathbf{0}(\mathbf{x}) = 0$ and $\mathbf{1}(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathbb{B}^n$;
- *negation* $\overline{\cdot} \colon \mathbb{B} \to \mathbb{B}$, $\overline{0} = 1$, $\overline{1} = 0$;
- *conjunction* $\wedge \colon \mathbb{B}^2 \to \mathbb{B}$, $x \wedge y = 1$ if and only if $x = y = 1$;
- *disjunction* $\vee \colon \mathbb{B}^2 \to \mathbb{B}$, $x \vee y = 0$ if and only if $x = y = 0$;
- *modulo*-2 *addition* $\oplus \colon \mathbb{B}^2 \to \mathbb{B}$, $x \oplus y = (x + y) \bmod 2$.

The set of all Boolean functions is denoted by Ω and the set of all projections is denoted by I_c. The preimage $f^{-1}(1)$ of 1 under f is referred to as the set of *true points*, while the preimage $f^{-1}(0)$ of 0 under f is referred to as the set of *false points*.

The *dual* of a Boolean function $f \colon \mathbb{B}^n \to \mathbb{B}$ is the function $f^{\mathrm{d}} \colon \mathbb{B}^n \to \mathbb{B}$ given by

$$f^{\mathrm{d}}(x_1, \dots, x_n) = \overline{f(\overline{x}_1, \dots, \overline{x}_n)}.$$

A variable x_i is *essential* in $f \colon \mathbb{B}^n \to \mathbb{B}$ if there are $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in \mathbb{B}$ such that

$$f(a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) \neq f(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n).$$

Variables that are not essential are said to be *inessential*.

Fact 1. *A variable x_i is essential in f if and only if it is essential in f^{d}.*

A Boolean function f is *self-dual* if $f = f^{\mathrm{d}}$.

1.3 Minors and relational constraints

A function $f \colon \mathbb{B}^m \to \mathbb{B}$ is a *minor* of another function $g \colon \mathbb{B}^n \to \mathbb{B}$ if there exists a map $\sigma \colon [n] \to [m]$ such that $f(\mathbf{a}) = g(\mathbf{a}\sigma)$ for all $\mathbf{a} \in \mathbb{B}^m$; in this case we write $f \leq g$. Functions f and g are *equivalent,* denoted $f \equiv g$, if $f \leq g$ and $g \leq f$. In other words, f is a minor of g if f can be obtained from g by permutation of arguments, addition and deletion of inessential arguments and identification of arguments. Functions f and g are equivalent if each one can be obtained from the other by permutation of arguments and addition and deletion of inessential arguments. The minor relation $\leq$ is a quasi-order (i.e., a reflexive and transitive relation) on the set of all Boolean functions, and the relation $\equiv$ is indeed an equivalence relation. For further background see, e.g., [5, 6, 7].

In what follows, we shall consider minors of a particular form. Let $f \colon \mathbb{B}^n \to \mathbb{B}$, let $i, j \in [n]$ ($i \neq j$). The function $f_{i=j} \colon \mathbb{B}^{n-1} \to \mathbb{B}$ given by

$$f_{i=j}(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n) = f(a_1, \dots, a_{i-1}, a_j, a_{i+1}, \dots, a_n),$$

for all $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in \mathbb{B}$, is called an *identification minor* of f.

Classes of functions that are closed under taking minors are known to be characterizable by so-called relational constraints. This was shown by Pippenger [19]. We will briefly survey some results which we will use hereinafter. An m-ary *relational constraint* is a couple (R, S) of m-ary relations R (the *antecedent*) and S (the *consequent*) on $\mathbb{B}$ (i.e., $R, S \subseteq \mathbb{B}^m$). We denote the antecedent and the consequent of a relational constraint Q by $R(Q)$ and $S(Q)$, respectively. The set of all relational constraints is denoted by Θ.

A function $f\colon \mathbb{B}^n \to \mathbb{B}$ *preserves* an m-ary relational constraint (R, S), denoted $f \triangleright (R, S)$, if for every $\mathbf{a}^1, \ldots, \mathbf{a}^n \in R$, we have $f(\mathbf{a}^1, \ldots, \mathbf{a}^n) \in S$. (Regarding tuples $\mathbf{a}^i$ as unary maps, $f(\mathbf{a}^1, \ldots, \mathbf{a}^n)$ denotes the m-tuple whose i-th entry is $f(\mathbf{a}^1, \ldots, \mathbf{a}^n)(i) = f(a_i^1, \ldots, a_i^n)$.)

The preservation relation gives rise to a Galois connection between functions and relational constraints that we briefly describe; for further background, see [2, 7, 19]. Define $\mathrm{cPol}\colon \mathcal{P}(\Theta) \to \mathcal{P}(\Omega)$, $\mathrm{cInv}\colon \mathcal{P}(\Omega) \to \mathcal{P}(\Theta)$ by

$$\mathrm{cPol}(\mathcal{Q}) = \{f \in \Omega : f \triangleright Q \text{ for every } Q \in \mathcal{Q}\},$$
$$\mathrm{cInv}(\mathcal{F}) = \{Q \in \Theta : f \triangleright Q \text{ for every } f \in \mathcal{F}\}.$$

We say that a set $\mathcal{F}$ of functions is *characterized* by a set $\mathcal{Q}$ of relational constraints if $\mathcal{F} = \mathrm{cPol}(\mathcal{Q})$. Dually, $\mathcal{Q}$ is *characterized* by $\mathcal{F}$ if $\mathcal{Q} = \mathrm{cInv}(\mathcal{F})$. In other words, sets of functions characterizable by relational constraints are exactly the fixed points of $\mathrm{cPol} \circ \mathrm{cInv}$, and, dually, sets of relational constraints characterizable by functions are exactly the fixed points of $\mathrm{cInv} \circ \mathrm{cPol}$.

The Galois connection cPol–cInv refines the well-known Galois connection Pol–Inv between functions and relations, which is induced by a similar preservation relation: function $f\colon \mathbb{B}^n \to \mathbb{B}$ *preserves* an m-ary relation R if for every $\mathbf{a}^1, \ldots, \mathbf{a}^n \in R$, we have $f(\mathbf{a}^1, \ldots, \mathbf{a}^n) \in R$. (In other words, f preserves a relation R if and only if f preserves the relational constraint (R, R).) Here, the Galois closed sets of functions coincide exactly with *clones*, i.e., classes of functions that contain all projections and that are closed under functional composition; for further background see, e.g., [8].

The following result reassembles various descriptions of the Galois closed sets of functions, which can be found in [7, 9, 19].

Theorem 2. *Let $\mathcal{F}$ be a set of functions. The following are equivalent.*

(i) *$\mathcal{F}$ is characterizable by relational constraints.*

(ii) *$\mathcal{F}$ is closed under taking minors.*

(iii) *$\mathcal{F}$ is of the form*

$$\mathrm{forbid}(A) := \{f \in \Omega : g \not\leq f \textit{ for all } g \in A\}$$

for some antichain A (i.e., set of pairwise incomparable functions) with respect to the minor relation $\leq$.

Remark 3. From the equivalence of (i) and (ii) in Theorem 2, it follows that the union and the intersection of classes that are characterizable by relational constraints are characterizable by relational constraints.

Remark 4. Note that the antichain A in Theorem 2 is unique up to equivalence. In fact, A can be chosen among the minimal elements of $\Omega \setminus \mathcal{F}$; the elements of A are called *minimal forbidden minors for $\mathcal{F}$*.

Remark 5. The Galois closed sets of relational constraints were likewise described in [19].

As we will see, there are classes of functions that, even though characterizable by relational constraints, are not characterized by any finite set of relational constraints. A set of functions is *finitely characterizable* if it is characterized by a finite set of relational constraints. The following theorem is a refinement of Theorem 2 and provides a description for finitely characterizable classes.

Theorem 6 ([7, 9]). *Let $\mathcal{F}$ be a set of functions. The following are equivalent.*

(i) *$\mathcal{F}$ is finitely characterizable.*

(ii) *$\mathcal{F}$ is of the form* $\mathrm{forbid}(A)$ *for some finite antichain A with respect to the minor relation $\leq$.*

2 Main results: classification and characterizations of Galois closed sets of threshold functions

2.1 Motivation

An n-ary Boolean function $f\colon \mathbb{B}^n \to \mathbb{B}$ is a *threshold function* if there are *weights* $w_1, \ldots, w_n \in \mathbb{R}$ and a *threshold* $t \in \mathbb{R}$, such that

$$f(x_1, \ldots, x_n) = 1 \iff \sum_{i=1}^{n} w_i x_i \geq t.$$

In other words, an n-ary Boolean function $f\colon \mathbb{B}^n \to \mathbb{B}$ is a threshold function if there is a hyperplane in $\mathbb{R}^n$ strictly seperating the true points of f from the false points of f. The set of all threshold functions is denoted by T.

The class of threshold functions has remarkable invariance properties. For instance, it is closed under taking negations and duals. Moreover, the class of threshold functions is also closed under taking minors of its members; hence it is characterizable by relational constraints by Theorem 2. However, it was shown by Hellerstein [11] that no finite set of relational constraints suffices.

Theorem 7. *The class of threshold functions is not finitely characterizable. Consequently, there exists an infinite antichain A such that for every $f \in A$, $f \notin T$ and $g \in T$ for every $g < f$.*

Imposing some additional conditions on threshold functions, we may obtain proper subclasses of T that are finitely characterizable. Easy examples arise from the intersections of T with the clone L of linear functions, the clone Λ of conjunctions or the clone V of disjunctions (see Appendix). However, other intersections may fail to be finitely characterizable. This fact gives rise to the following problem.

Problem. Which clones C of Boolean functions have the property that $C \cap T$ is finitely characterizable?

In the following subsection we present a solution to this problem.

2.2 Complete classification and corresponding characterizations of subclasses of threshold functions

For any clone C contained in one of L, V and Λ, the intersection $C \cap T$ is a clone. For,

$$L \cap T = \Omega(1), \qquad \Lambda \subseteq T, \qquad V \subseteq T.$$

Hence, the characterization of $C \cap T$ for any clone C contained in one of L, V and Λ is given by relational constraints of the form (R, R) for those relations R characterizing C (as given in the Appendix).

We proceed to characterizing the remaining subclasses $T \cap C$ of threshold functions that, as we will see, are not finitely characterizable.

A characterization of the class T of threshold functions (i.e., for $C = \Omega$), is given by the following family of relational constraints. Define for $n \geq 1$, the $2n$-ary relational constraint B_n as

$$R(B_n) := \{(x_1, \ldots, x_{2n}) \in \mathbb{B}^{2n} : \sum_{i=1}^{n} x_i = \sum_{i=n+1}^{2n} x_i\},$$

$$S(B_n) := \mathbb{B}^{2n} \setminus \{(\underbrace{0,\ldots,0}_{n},\underbrace{1,\ldots,1}_{n}), (\underbrace{1,\ldots,1}_{n},\underbrace{0,\ldots,0}_{n})\}.$$

(Note that in the definition of $R(B_n)$ we employ the usual addition of real numbers.)

Theorem 8. *The set* $\mathrm{cPol}\{B_n : n \geq 2\}$ *is the class of all threshold functions.*

Moreover, for every clone C*, the subclass* $C \cap T$ *of threshold functions is characterized by the set* $\{B_n : n \geq 1\} \cup \mathcal{R}_C$*, where* $\mathcal{R}_C$ *is the set of relational constraints characterizing the clone* C*, as given in the Appendix.*

Theorem 8 provides an infinite set of relational constraints characterizing the set $C \cap T$ for each clone C. As the following classification reveals, the characterization provided is optimal for the clones not contained in L, V or Λ in the sense that for such clones C, the class $C \cap T$ is not finitely characterizable by relational constraints.

Theorem 9. *Let* C *be a clone of Boolean functions. The subclass* $C \cap T$ *of threshold functions is finitely characterizable if and only if* C *is contained in one of the clones* L*,* V*,* Λ*.*

Appendix. Description of Boolean clones and corresponding characterizing sets of relational constraints

We provide a concise description of all Boolean clones as well as characterizing sets of relations R – or, equivalently, relational constraints (R, R) – for some clones; the characterization of the remaining clones is easily derived by noting that if $C_1 = \mathrm{cPol}(\mathcal{Q}_1)$ and $C_2 = \mathrm{cPol}(\mathcal{Q}_2)$, then $C_1 \cap C_2 = \mathrm{cPol}(\mathcal{Q}_1 \cup \mathcal{Q}_2)$. We make use of notations and terminology appearing in [10] and [13].

- Ω denotes the clone of all Boolean functions. It is characterized by the empty relation.

- T_0 and T_1 denote the clones of 0- and 1-preserving functions, respectively, i.e.,

$$T_0 = \{f \in \Omega : f(0,\ldots,0) = 0\},$$
$$T_1 = \{f \in \Omega : f(1,\ldots,1) = 1\}.$$

They are characterized by the unary relations $\{0\}$ and $\{1\}$, respectively.

- T_c denotes the clone of constant-preserving functions, i.e., $T_c = T_0 \cap T_1$.

- M denotes the clone of all monotone functions, i.e.,

$$M = \{f \in \Omega : f(\mathbf{a}) \leq f(\mathbf{b}) \text{ whenever } \mathbf{a} \leq \mathbf{b}\}.$$

It is characterized by the binary relation

$$\leq := \{(0,0),(0,1),(1,1)\}.$$

- $M_0 = M \cap T_0$, $M_1 = M \cap T_1$, $M_c = M \cap T_c$.

- S denotes the clone of all self-dual functions, i.e.,

$$S = \{f \in \Omega : f^{\mathrm{d}} = f\}.$$

It is characterized by the binary relation

$$\{(0,1),(1,0)\}.$$

- $S_c = S \cap T_c$, $SM = S \cap M$.

- L denotes the clone of all linear functions, i.e.,

$$L = \{f \in \Omega : f = c_0 \oplus c_1 x_1 \oplus \cdots \oplus c_n x_n\}.$$

It is characterized by the quaternary relation

$$\{(a,b,c,d) \,:\, a \oplus b \oplus c = d\}.$$

- $L_0 = L \cap T_0$, $L_1 = L \cap T_1$, $LS = L \cap S$, $L_c = L \cap T_c$.

Let $a \in \{0,1\}$. A set $A \subseteq \{0,1\}^n$ is said to be *a-separating* if there is some $i \in [n]$ such that for every $(a_1,\ldots,a_n) \in A$ we have $a_i = a$. A function f is said to be *a-separating* if $f^{-1}(a)$ is a-separating. The function f is said to be *a-separating of rank* $k \geq 2$ if every subset $A \subseteq f^{-1}(a)$ of size at most k is a-separating.

• For $m \geq 2$, U_m and W_m denote the clones of all 1- and 0-separating functions of rank m, respectively. They are characterized by the m-ary relations $\mathbb{B}^m \setminus \{(0,\dots,0)\}$ and $\mathbb{B}^m \setminus \{(1,\dots,1)\}$, respectively.

• U_∞ and W_∞ denote the clones of all 1- and 0-separating functions, respectively, i.e., $U_\infty = \bigcap_{k\geq 2} U_k$ and $W_\infty = \bigcap_{k\geq 2} W_k$.

• $T_cU_m = T_c \cap U_m$ and $T_cW_m = T_c \cap W_m$, for $m = 2,\dots,\infty$.

• $MU_m = M \cap U_m$ and $MW_m = M \cap W_m$, for $m = 2,\dots,\infty$.

• $M_cU_m = M_c \cap U_m$ and $M_cW_m = M_c \cap W_m$, for $m = 2,\dots,\infty$.

• Λ denotes the clone of all conjunctions and constants, i.e.,

$$\Lambda = \{f \in \Omega : f = x_{i_1} \wedge \cdots \wedge x_{i_n}\} \cup \{\mathbf{0}^{(n)} : n \geq 1\} \cup \{\mathbf{1}^{(n)} : n \geq 1\}.$$

It is characterized by the ternary relation

$$\{(a,b,c) \,:\, a \wedge b = c\}.$$

• $\Lambda_0 = \Lambda \cap T_0$, $\Lambda_1 = \Lambda \cap T_1$, $\Lambda_c = \Lambda \cap T_c$.

• V denotes the clone of all disjunctions and constants, i.e.,

$$V = \{f \in \Omega : f = x_{i_1} \vee \cdots \vee x_{i_n}\} \cup \{\mathbf{0}^{(n)} : n \geq 1\} \cup \{\mathbf{1}^{(n)} : n \geq 1\}.$$

It is characterized by the ternary relation

$$\{(a,b,c) \,:\, a \vee b = c\}.$$

• $V_0 = V \cap T_0$, $V_1 = V \cap T_1$, $V_c = V \cap T_c$.

• $\Omega(1)$ denotes the clone of all projections, negations, and constants. It is characterized by the ternary relation

$$\{(a,b,c) \,:\, a = b \quad \text{or} \quad b = c\}.$$

• $I^* = \Omega(1) \cap S$, $I = \Omega(1) \cap M$.

• $I_0 = I \cap T_0$, $I_1 = I \cap T_1$.

• I_c denotes the smallest clone containing only projections, i.e., $I_c = I \cap T_c$.

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